

# THE STABILITY OF STRONG VISCOUS CONTACT DISCONTINUITY TO AN INFLOW PROBLEM FOR FULL COMPRESSIBLE NAVIER-STOKES EQUATIONS

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**Abstract.** This paper is concerned with nonlinear stability of viscous contact discontinuity to inflow problem for the one-dimensional full compressible Navier-Stokes equations with different ends in half space  $[0, \infty)$ . For the case when the local stability of the contact discontinuities was first studied by [1], later generalized by [2], local stability of weak viscous contact discontinuity is well-established by [4–8], but for the global stability of inflow gas with big oscillation ends ( $|\theta_+ - \theta_-| > 1$  and  $|\rho_+ - \rho_-| > 1$ ), fewer results have been obtained excluding zero dissipation [9] or  $\gamma \rightarrow 1$  gas see [10]. Our main purpose is to deduce the corresponding nonlinear stability result with the two different ends by exploiting the elementary energy method. As a first step towards this goal, we will show in this paper that with a certain class of big perturbation which can allow  $|\theta_- - \theta_+| > 1$  and  $|\rho_- - \rho_+| > 1$ , the global stability result holds.

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## 1 Introduction

This paper is concerned with an “inflow problem” for a one-dimensional compressible viscous heat-conducting flow in the half space  $\mathbb{R}_+ = [0, \infty)$ , which is governed by the following initial-boundary value problem in Eulerian coordinate  $(\tilde{x}, t)$ :

$$\begin{cases} \tilde{\rho}_t + (\tilde{\rho}\tilde{u})_{\tilde{x}} = 0, & (\tilde{x}, t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ (\tilde{\rho}\tilde{u})_t + (\tilde{\rho}\tilde{u}^2 + \tilde{p})_{\tilde{x}} = \mu\tilde{u}_{\tilde{x}\tilde{x}}, \\ \left( \tilde{\rho} \left( \tilde{e} + \frac{\tilde{u}^2}{2} \right) \right)_t + \left( \tilde{\rho}\tilde{u} \left( \tilde{e} + \frac{\tilde{u}^2}{2} \right) + \tilde{p}\tilde{u} \right)_{\tilde{x}} = \kappa\tilde{\theta}_{\tilde{x}\tilde{x}} + (\mu\tilde{u}\tilde{u}_{\tilde{x}})_{\tilde{x}}, \\ (\tilde{\rho}, \tilde{u}, \tilde{\theta})|_{\tilde{x}=0} = (\rho_-, u_b, \theta_-) \quad \text{with} \quad u_b > 0, \\ (\tilde{\rho}, \tilde{u}, \tilde{\theta})|_{t=0} = (\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0)(\tilde{x}) \rightarrow (\rho_+, u_b, \theta_+) \quad \text{as} \quad \tilde{x} \rightarrow \infty, \end{cases} \quad (1.1)$$

where  $\tilde{\rho}$ ,  $\tilde{u}$  and  $\tilde{\theta}$  are the density, the velocity and the absolute temperature, respectively, while  $\mu > 0$  is the viscosity coefficient and  $\kappa > 0$  is the heat-conductivity coefficients, respectively. It is assumed throughout the paper that  $\rho_{\pm}$ ,  $u_b$  and  $\theta_{\pm}$  are prescribed positive constants. We shall

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focus our interest on the polytropic ideal gas with  $|\theta_+ - \theta_-|$  and  $|\rho_+ - \rho_-|$  are general constants (not small), so the pressure  $\tilde{p} = \tilde{p}(\tilde{\rho}, \tilde{\theta})$  and the internal energy  $\tilde{e} = \tilde{e}(\tilde{\rho}, \tilde{\theta})$  are related by the second law of thermodynamics:

$$\tilde{p} = R\tilde{\rho}\tilde{\theta}, \quad \tilde{e} = \frac{R}{\gamma - 1}\tilde{\theta} + \text{const.}, \quad (1.2)$$

where  $\gamma > 1$  is the adiabatic exponent and  $R > 0$  is the gas constant.

The boundary condition  $(1.1)_4$  implies that, through the boundary  $\tilde{x} = 0$ , the fluid with density  $\rho_-$  flows into the region  $\mathbb{R}_+$  at the speed  $u_b > 0$ . So the initial-boundary value problem (1.1) is the so-called *inflow* problem. On the other hand, in the case that  $u_b = 0$  (resp.  $u_b < 0$ ), the problem is called the *impermeable wall* (resp. *outflow*) problem in which the boundary condition of density can't be imposed. In terms of various boundary values, Matsumura [11] classified all possible large-time behaviors of the solutions for the one-dimensional (isentropic) compressible Navier-Stokes equations.

Our main purpose is to study the asymptotic stability of the contact discontinuity for the inflow problem (1.1). It is well known that there are three basic wave patterns for the 1D compressible Euler equations, including two nonlinear waves, say shock and rarefaction waves, and a linearly degenerate wave, say contact discontinuity. There have been a lot of works on the asymptotic behaviors of solutions to the initial-boundary value (or Cauchy) problem for the Navier-Stokes equations toward these basic waves or their viscous versions, see, for example, [3–26] and the reference therein. In what follows, we briefly recall some related references. Concerning the inflow problem, Matsumura and Nishihara [18] considered an inflow problem for the one-dimensional isentropic model system of compressible viscous gas (i.e. the 1D isentropic Navier-Stokes  $(1.1)_1 - (1.1)_2$  with  $\tilde{p} = R\tilde{\rho}^\gamma$ ) and established the stability theorems on both the boundary layer solution and the superposition of a boundary-layer solution and a rarefaction wave. We also refer to the paper due to Huang et al. [3] in which the asymptotic stability on both the viscous shock wave and the superposition of a viscous shock wave and a boundary-layer solution are studied. On the other hand, the problem of stability of contact discontinuities are associated with linear degenerate fields and are less stable than the nonlinear waves for the inviscid system (Euler equations). It was observed in [1, 2], where the metastability of contact waves was studied for viscous conservation laws with artificial viscosity, that the contact discontinuity cannot be the asymptotic state for the viscous system, and a diffusive wave, which approximated the contact discontinuity on any finite time interval, actually dominates the large-time behavior of solutions. The nonlinear stability of contact discontinuity for the (full) compressible Navier-Stokes equations was then investigated in [4, 7] for the free boundary value problem and [5, 6] for the Cauchy problem.

However, to our best knowledge, fewer mathematical literature known for the large-time behaviors of solutions to the inflow problem of the full compressible viscous heat-conducting Navier-Stokes equations due to various difficulties come from the big oscillation ends. So the aim of this paper is to show that the contact discontinuities are metastability wave patterns for the inflow problem (1.1) of the full Navier-Stokes system.

To state our main results we first transfer (1.1) to the problem in the Lagrangian coordinate and then make use of a coordinate transformation to reduce the initial-boundary value problem

(1.1) into the following form:

$$\begin{cases} v_t - sv_x - u_x = 0, & (x, t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ u_t - su_x + \left(\frac{R\theta}{v}\right)_x = \mu \left(\frac{u_x}{v}\right)_x, \\ \frac{R}{\gamma-1}\theta_t - \frac{R}{\gamma-1}s\theta_x + R\frac{\theta}{v}u_x = \kappa \left(\frac{\theta_x}{v}\right)_x + \mu \frac{u_x^2}{v}, \\ (v, u, \theta)|_{x=0} = (v_-, u_b, \theta_-), \quad t > 0, \\ (v, u, \theta)|_{t=0} = (v_0, u_0, \theta_0) \rightarrow (v_+, u_b, \theta_+) \quad \text{as } x \rightarrow \infty, \end{cases} \quad (1.3)$$

where  $v_{\pm}$ ,  $u_b$  and  $\theta_{\pm}$  are given positive constants, and  $s = -u_b/v_- < 0$ ,  $v_0, \theta_0 > 0$ . In fact  $v = 1/\rho(x, t)$ ,  $u = u(x, t)$ ,  $\theta = \theta(x, t)$  and  $R\theta/v = p(v, \theta)$  are the specific volume, velocity, temperature and pressure as in (1.1).

Recently most of these Navier-Stokes equations use their Euler systems as their limitations. Here we consider the corresponding Euler system of (1.3) with Riemann initial data reads as follows:

$$\begin{cases} v_t - sv_x - u_x = 0, \\ u_t - su_x + p(v, \theta)_x = 0, \\ \frac{R}{\gamma-1}\theta_t - \frac{R}{\gamma-1}s\theta_x + R\frac{\theta}{v}u_x = 0, \\ (v, u, \theta)(x, 0) = (v_-, u_b, \theta_-) \quad \text{if } x < 0, \\ (v, u, \theta)(x, 0) = (v_+, u_b, \theta_+) \quad \text{if } x > 0, \end{cases} \quad (1.4)$$

where  $v_{\pm} = \frac{1}{\rho_{\pm}}$ ,  $u_b$  and  $\theta_{\pm}$  are the same positive constants as in (1.1).

Because the corresponding Euler equations (1.4) with the Riemann initial data has the following solitons

$$(\overline{V}, \overline{U}, \overline{\Theta}) = \begin{cases} (v_-, u_b, \theta_-), & x < -st, \\ (v_+, u_b, \theta_+), & x > -st, \end{cases} \quad (1.5)$$

provided that

$$p_- = R\frac{\theta_-}{v_-} = p_+ = R\frac{\theta_+}{v_+}. \quad (1.6)$$

As that in [4] we conjecture that the asymptotic limit  $(V, U, \Theta)$  of (1.3) is as follows

$$P(V, \Theta) = R\frac{\Theta}{V} = p_+, \quad U(x, t) = \frac{\kappa(\gamma-1)\Theta_x}{\gamma R\Theta} + u_b, \quad (1.7)$$

and  $\Theta$  is the solution of the following problem

$$\begin{cases} \Theta_t - s\Theta_x = a(\ln \Theta)_{xx}, & a = \frac{\kappa p_+(\gamma-1)}{\gamma R^2} > 0, \\ \Theta(0, t) = \theta_-, \\ \Theta(x, 0) = \Theta_0 \rightarrow \theta_+. \end{cases} \quad (1.8)$$

$(V, U, \Theta)$  satisfies

$$\left\{ \begin{array}{l} R \frac{\Theta}{V} = p_+, \\ V_t - sV_x = U_x, \\ U_t - sU_x + P(V, \Theta)_x = \mu \left( \frac{U_x}{V} \right)_x + F, \\ \frac{R}{\gamma-1} \Theta_t - s \frac{R}{\gamma-1} \Theta_x + R \frac{\Theta}{V} U_x = \kappa \left( \frac{\Theta_x}{V} \right)_x + \mu \frac{U_x^2}{V} + G, \\ (V, U, \Theta)(0, t) = (v_-, \frac{\kappa(\gamma-1)}{\gamma R} \frac{\Theta_x}{\Theta}|_{x=0} + u_b, \theta_-), \\ (V, U, \Theta)(x, 0) = (V_0, U_0, \Theta_0) = (\frac{R}{p_+} \Theta_0, \frac{\kappa(\gamma-1)}{\gamma R} \frac{\Theta_{0x}}{\Theta_0} + u_b, \Theta_0) \rightarrow (v_+, u_b, \theta_+), \text{ as } x \rightarrow +\infty, \end{array} \right. \quad (1.9)$$

where

$$\begin{aligned} G &= -\mu \frac{U_x^2}{V} = O((\ln \Theta)_{xx}^2), \\ F(x, t) &= \frac{\kappa(\gamma-1)}{\gamma R} \left\{ (\ln \Theta)_{xt} - s(\ln \Theta)_{xx} - \mu \left( \frac{(\ln \Theta)_{xx}}{V} \right)_x \right\} \\ &= \frac{\kappa a(\gamma-1) - \mu p_+ \gamma}{R \gamma} \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right)_x. \end{aligned} \quad (1.10)$$

Denote

$$\begin{aligned} \varphi(x, t) &= v(x, t) - V(x, t), \\ \psi(x, t) &= u(x, t) - U(x, t), \\ \zeta(x, t) &= \theta(x, t) - \Theta(x, t). \end{aligned} \quad (1.11)$$

Combining (1.9) and (1.3), the original problem can be reformulated as

$$\left\{ \begin{array}{l} \varphi_t - s\varphi_x = \psi_x, \\ \psi_t - s\psi_x - \left( \frac{R\Theta}{vV} \varphi \right)_x + \left( \frac{R\zeta}{v} \right)_x = -\mu \left( \frac{U_x}{vV} \varphi \right)_x + \mu \left( \frac{\psi_x}{v} \right)_x - F, \\ \frac{R}{\gamma-1} \zeta_t - s \frac{R}{\gamma-1} \zeta_x + \frac{R\theta}{v} (\psi_x + U_x) - \frac{R\Theta}{V} U_x = \kappa \left( \frac{\zeta_x}{v} \right)_x - \kappa \left( \frac{\Theta_x \varphi}{vV} \right)_x + \mu \left( \frac{u_x^2}{v} - \frac{U_x^2}{V} \right) - G, \\ (\varphi, \psi, \zeta)(0, t) = (0, u_b - U(0, t), 0), \\ (\varphi, \psi, \zeta)(x, 0) = (\varphi_0, \psi_0, \zeta_0) = (v_0 - V_0, u_0 - U_0, \theta_0 - \Theta_0). \end{array} \right. \quad (1.12)$$

Under the above preparation in hand, our original problem can be transferred into a stability problem: If the initial data  $(v_0(x), u_0(x), \theta_0(x))$  of the inflow problem (1.1) admit a unique global solution  $(v(x, t), u(x, t), \theta(x, t))$  which tend to  $(V(x, t), U(x, t), \Theta(x, t))$  as  $t \rightarrow \infty$ ? Recall that according to whether  $H(\mathbb{R}_+)$ -norm of the initial perturbation  $(\varphi_0(x), \psi_0(x), \zeta_0(x))$  and (or)  $|(\theta_+ - \theta_-, \rho_+ - \rho_-)| > 1$  or not, the stability results are classified into global (or local) stability of strong (or weak) viscous contact wave.

To deduce the desired nonlinear stability result by the elementary energy method as in [3-8, 10, 12, 13], it is sufficient to deduce certain uniform (with respect to the time variable

$t$ ) energy type estimates on the solution  $(v(x, t), u(x, t), \theta(x, t))$  and the main difficulty to do so lies in how to deal with the boundary condition when we get rid off the small condition of  $|\theta_+ - \theta_-|$  and how to establish the Poincaré type inequality in Lemma 3.2 without the smallness of  $|\theta_+ - \theta_-|$  which the arguments employed in [3–8, 12, 13] is to use both smallness  $|\theta_+ - \theta_-|$  and  $N(t) = \sup_{0 \leq \tau \leq t} \|(\varphi, \psi, \zeta)\|_{H^1}$  to overcome such difficulties. One of the key points in such an argument is that, based on the a priori assumption that  $\sup_{0 \leq \tau \leq t} \|(\varphi, \psi, \zeta)\|_{H^1}(\tau)$  is sufficiently small, one can deduce a uniform lower and upper positive bounds on the specific volume  $v(x, t)$  and temperature  $\theta(x, t)$ . With such a bound on  $v$  and  $\theta$  in hand, one can deduce certain a priori  $H(\mathbb{R}_+)$  energy type estimates on  $(\varphi, \psi, \zeta)$  in terms of the initial perturbation  $(\varphi_0, \psi_0, \zeta_0)$  provided that  $\|\Theta_{0x}\|$  suitably small, so stability of weak contact discontinuity can be obtained. In fact if  $N(t)$  not small and the perturbation of  $\|(\varphi_{0x}, \psi_{0x}, \zeta_{0x})\|_{L^2(\mathbb{R}_+)}$  not small (see [10]), the combination of the analysis similar as above with the standard continuation argument, we can obtain the upper and lower bounds of  $(v, \theta)$ , then that yields the global stability of strong viscous contact discontinuity for the one-dimensional compressible Navier-Stokes equations. So it is important to finish a priori estimate without the smallness of  $|\theta_+ - \theta_-|$  and  $\|(\varphi_{0x}, \psi_{0x}, \zeta_{0x})\|_{L^2(\mathbb{R}_+)}$ .

This paper we replace self-similar solution (see [3–8]) to a diffusion equation's solution. We use the fundamental solution skill in [13] and give some precise time estimates about temperature  $\Theta$  which can cause to the global uniform time estimate, so the similar energy priori estimate as the refers can be obtained. The global stability result comes out because of these time estimates. It is easy to see that in such a result, for all  $t \in \mathbb{R}_+$ ,  $\text{Osc } \theta(t) := \sup_{x \in \mathbb{R}_+} \theta(x, t) - \inf_{x \in \mathbb{R}_+} \theta(x, t) \geq |\theta_+ - \theta_-|$ , the oscillation of the temperature  $\theta(x, t)$  should not be sufficiently small.

To state our main result, we assume throughout of this section that

$$(\varphi_0, \zeta_0)(x) \in H_0^1(0, \infty), \quad \psi_0(x) \in H^1(0, \infty).$$

Moreover, for an interval  $I \in [0, \infty)$ , we define the function space

$$X(I) = \{(\varphi, \psi, \zeta) \in C(I, H^1) | \varphi_x \in L^2(I; L^2), (\psi_x, \zeta_x) \in L^2(I; H^1)\}.$$

Our main results of this paper now reads as follows.

**Theorem 1.1** *There exist positive constants  $C, \alpha, \delta_0$  and  $\eta_0$  such that if  $1 < |\theta_+ - \theta_-|$ ,  $\delta_0$  independent of  $\theta_{\pm}$ ,*

$$\Theta_0 = \theta_+ - (\theta_+ - \theta_-) \exp\{1 - (1 + \alpha x)^{\delta_0}\},$$

*and  $\|(v_0 - V_0, u_0 - U_0, \theta_0 - \Theta_0)\|_{L^2} \leq \eta_0$ ,  $\|(v_{0x} - V_{0x}, u_{0x} - U_{0x}, \theta_{0x} - \Theta_{0x})\|_{L^2} \leq C$ , (1.12) has a unique global solution  $(\varphi, \psi, \zeta)$  satisfying  $(\varphi, \psi, \zeta) \in X([0, \infty))$  and*

$$\sup_{x \in \mathbb{R}_+} |(\varphi, \psi, \zeta)| \rightarrow 0, \text{ as } t \rightarrow \infty.$$

**Remark 1.1** *The constant  $\alpha$  will be determined in Lemma 2.4 for the definition of viscous contact discontinuity in [4], which is on any finite-time interval as  $k \rightarrow 0$ ,  $(V, U, \Theta)$  is a viscous contact wave when  $\|(V - \bar{V}, U - \bar{U}, \Theta - \bar{\Theta})\|_{L^p} \rightarrow 0$ .*

## 2 Preliminary

In this section, to study the asymptotic behavior of the solution to inflow problem (1.3), we will do some preparations lemmas and list a priori estimate which are important to the proof of Theorem 1.1.

Throughout this paper, we shall denote  $H^l(\mathbb{R}_+)$  the usual  $l$ -th order Sobolev space with the norm

$$\|f\|_l = \left( \sum_{j=0}^l \|\partial_x^j f\|^2 \right)^{1/2}, \quad \|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}_+)}.$$

For simplicity, we also use  $C$  or  $C_i$  ( $i = 1, 2, 3, \dots$ ) to denote the various positive generic constants.  $C(z)$  stands for constant about  $z$  and  $\lim_{z \rightarrow 0} C(z) = 0$ .  $\epsilon$  and  $\epsilon_i$  ( $i = 1, 2, 3, \dots$ ) stand for suitably

small positive constant in Cauchy-Schwarz inequality and  $\partial_x^i = \frac{\partial^i}{\partial x^i}$ .

We shall prove Theorem 1.1 by combining the local existence and the global-in-time a priori estimates. Since the local existence of the solution is well known (see, for example, [4]), we omit it here for brevity. to prove the global existence part of Theorem 1.1, it is sufficient to establish the following a priori estimates.

**Proposition 2.1** (A priori estimate) *Let  $(\varphi, \psi, \zeta) \in X([0, t])$  be a solution of problem (1.12) for some  $t > 0$ . Then there exist positive constants  $C(\delta_0) < 1$  and  $C$  which are all independent of  $t$  and  $(v, \theta)$ , such that if  $m \leq v, \theta \leq M$  and  $N(t) = \sup_{0 \leq \tau \leq t} \|(\varphi, \psi, \zeta)\|_1 \leq C$ , it holds that*

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \|(\psi, \varphi, \zeta)\|^2(t) + \int_0^t \|(\psi_x, \zeta_x)\|^2(\tau) d\tau \\ & \leq C\|(\varphi_0, \psi_0, \zeta_0)\| + C(\delta_0). \\ & \sup_{0 \leq \tau \leq t} \|(\psi_x, \varphi_x, \zeta_x)\|^2(t) + \int_0^t (\|\varphi_x\|^2(\tau) + \|(\psi_x, \zeta_x)\|_1^2(\tau)) d\tau \\ & \leq C\|(\varphi_0, \psi_0, \zeta_0)\|_1 + C(\delta_0). \end{aligned} \tag{2.1}$$

To finish this proposition, we must consider some properties of  $\Theta_0$  and  $\partial_x^i \Theta$  ( $i = 1, 2, 3, \dots$ ) as we list following.

**Lemma 2.1** *As to the definition of  $\Theta_0$  in Theorem 1.1 we have*

$$\begin{aligned} \|\Theta_0 - \theta_+\|_{L^1} & \leq C\alpha^{-1} \sum_{n=0}^{\lfloor \frac{1}{\delta_0} \rfloor - 1} \prod_{i=0}^n \left( \frac{1}{\delta_0} - i \right), \\ 0 < \Theta_{0x} & \leq C\alpha\delta_0(1 + \alpha x)^{\delta_0 - 1} \exp\{-(1 + \alpha x)^{\delta_0}\}, \\ |\Theta_{0xx}| & \leq C\alpha^2\delta_0((1 + \alpha x)^{2\delta_0 - 2} + (1 - \delta_0)(1 + \alpha x)^{\delta_0 - 2}) \exp\{-(1 + \alpha x)^{\delta_0}\}, \\ \|\Theta_{0x}\|^2 & \leq C\alpha\delta_0, \\ \|\Theta_{0x}\|_{L^1(\mathbb{R}_+)} & \leq C, \\ \|\Theta_{0xx}\|^2 + \|(\ln \Theta_0)_{xx}\|^2 & \leq C\alpha^3\delta_0^2, \\ \|\Theta_{0xxx}\|^2 + \|(\ln \Theta_0)_{xxx}\|^2 & \leq C, \\ \int_{\mathbb{R}} \Theta_{0x}^2(1 + \alpha x) dx & \leq C\alpha\delta_0. \end{aligned}$$

*Proof.* In fact

$$\begin{aligned}
\int_{\mathbb{R}_+} |\Theta_0 - \theta_+| dx &\leq C\alpha^{-1} \int_{\mathbb{R}_+} \exp\{-(1+\alpha x)^{\delta_0}\} d\left((\alpha x + 1)^{\delta_0}\right)^{1/\delta_0} \\
&= C\alpha^{-1} \int_0^{+\infty} \exp\{-(1+\alpha x)^{\delta_0}\} \frac{1}{\delta_0} \left((1+\alpha x)^{\delta_0}\right)^{1/\delta_0-1} d(\alpha x + 1)^{\delta_0} \\
&= C\alpha^{-1} \frac{1}{\delta_0} \int_1^{+\infty} \exp\{-z\} z^{1/\delta_0-1} dz \\
&= C\alpha^{-1} \frac{1}{\delta_0} \exp\{-z\} z^{1/\delta_0-1} \Big|_{+\infty}^1 + C\alpha^{-1} \frac{1}{\delta_0} \int_1^{+\infty} \exp\{-z\} dz^{1/\delta_0-1} \\
&= C\alpha^{-1} \frac{1}{\delta_0} \exp\{-z\} z^{1/\delta_0-1} \Big|_{+\infty}^1 + \dots + C\alpha^{-1} \prod_{i=0}^{\lceil \frac{1}{\delta_0} \rceil - 1} \left(\frac{1}{\delta_0} - i\right) \int_1^{+\infty} \exp\{-z\} z^{1/\delta_0 - \lceil 1/\delta_0 \rceil} dz \\
&\leq C\alpha^{-1} \sum_{n=0}^{\lceil \frac{1}{\delta_0} \rceil - 1} \prod_{i=0}^n \left(\frac{1}{\delta_0} - i\right).
\end{aligned}$$

That is the first inequality.

As to the last inequality, we know that

$$\begin{aligned}
\int_{\mathbb{R}} |\Theta_{0x}|^2 (1+\alpha x) dx &\leq \int_{\mathbb{R}} C\alpha^2 \delta_0^2 (1+\alpha x)^{2\delta_0-2} (1+\alpha x) \exp\{-2(1+\alpha x)^{\delta_0}\} dx \\
&= \int_{\mathbb{R}} C\alpha^2 \delta_0^2 (1+\alpha x)^{2\delta_0-1} \exp\{-2(1+\alpha x)^{\delta_0}\} dx \\
&= \int_{\mathbb{R}} C\alpha \delta_0^2 (2\delta_0)^{-1} \exp\{-2(1+\alpha x)^{\delta_0}\} d(1+\alpha x)^{2\delta_0} \\
&= \int_{\mathbb{R}} C2^{-1} \alpha \delta_0 \exp\{-2z\} dz^2 \leq C\alpha \delta_0.
\end{aligned}$$

The other inequalities can be check easily by using the definition of  $\Theta_0$  and we omit to write them.  $\square$

Next, we construct a parabolic equation about  $\theta_2$ , it will be used in the estimates of  $\partial_x^i \Theta$ .

**Lemma 2.2** *If  $\delta_0$  and  $\Theta_0$  satisfying the condition in Theorem 1.1 and*

$$\begin{aligned}
\theta_2(x, t) &= \int_0^{+\infty} (4\pi at)^{-1/2} (\Theta_0(h) - \theta_-) \left\{ \exp\left\{-\frac{(h-st-x)^2}{4at}\right\} - \exp\left\{-\frac{(h-st+x)^2}{4at}\right\} \right\} dh + \theta_-, \\
K &= \int_0^{+\infty} (4\pi at)^{-1/2} \Theta_{0z}(z) \exp\left\{-\frac{(z-st+x)^2}{4at}\right\} dz,
\end{aligned}$$

we can get

$$\begin{aligned}
\theta_{2t} - s\theta_{2x} &= a\theta_{2xx} - 2sK; \\
\theta_2(0, t) &= \theta_-; \\
\theta_2(x, 0) = \theta_{20}(x) &= \begin{cases} \Theta_0(x) \rightarrow \theta_+, & x > 0; \\ -\Theta_0(-x) + 2\theta_- \rightarrow 2\theta_- - \theta_+, & x \leq 0, \end{cases} \quad (2.2)
\end{aligned}$$

and

$$\int_0^t \|K\|_{L^1(\mathbb{R}_+)} dt \leq C\alpha^{1/2}\delta_0^{1/2}(1+t)^{1/2}, \quad (2.3)$$

$$\int_0^t \|\theta_{2x}\|^2 dt \leq C(1+t)^{1/2}, \quad (2.4)$$

$$\|\theta_{2x}\|^2 + \int_0^t \theta_{2x}^2(0,t) dt + \int_0^t \int_0^{+\infty} \theta_{2xx}^2 dx dt \leq C\alpha\delta_0. \quad (2.5)$$

*Proof.* First we proof (2.2)<sub>1</sub> as following:

Because

$$\begin{aligned} \theta_{2x} &= \int_0^{+\infty} (4\pi at)^{-1/2} (\Theta_0(z) - \theta_-) \exp\left\{-\frac{(z-st-x)^2}{4at}\right\} \frac{z-st-x}{2at} dz \\ &\quad + \int_0^{+\infty} (4\pi at)^{-1/2} (\Theta_0(z) - \theta_-) \exp\left\{-\frac{(z-st+x)^2}{4at}\right\} \frac{z-st+x}{2at} dz \\ &= \int_0^{+\infty} (4\pi at)^{-1/2} \Theta_{0z}(z) \exp\left\{-\frac{(z-st-x)^2}{4at}\right\} dz \\ &\quad - \int_0^{+\infty} (4\pi at)^{-1/2} \Theta_{0z}(z) \exp\left\{-\frac{(z-st+x)^2}{4at}\right\} dz \\ &=: \hat{I}_1 + \hat{I}_2, \end{aligned} \quad (2.6)$$

it is easy to check  $\theta_{2t} - s\theta_{2x} = a\theta_{2xx} - 2sK$ ,  $K = -\hat{I}_2$  and we finish (2.2)<sub>1</sub> and (2.2)<sub>2</sub>.

Now we proof (2.2)<sub>3</sub> as following.

From heat conduction equation's initial theorem and uniform estimates we know that

$$\begin{aligned} \theta_{20}(x) &= \lim_{t \rightarrow 0} \int_{-\infty}^{+\infty} (4\pi at)^{-1/2} \theta_{20}(h) \exp\left\{-\frac{(h-x)^2}{4at}\right\} dh \\ &= \lim_{t \rightarrow 0} \int_0^{+\infty} (4\pi at)^{-1/2} (\Theta_0(h) - \theta_-) \exp\left\{-\frac{(h-x)^2}{4at}\right\} dh \\ &\quad + \lim_{t \rightarrow 0} \int_0^{+\infty} (4\pi at)^{-1/2} (-\Theta_0(h) + \theta_-) \exp\left\{-\frac{(h+x)^2}{4at}\right\} dh + \theta_-. \end{aligned}$$

So

$$\begin{aligned} \lim_{t \rightarrow 0} (\theta_2(x,t) - \theta_{20}(x)) &= \lim_{t \rightarrow 0} \int_0^{+\infty} (4\pi at)^{-1/2} (\Theta_0(h) - \theta_-) \left( \exp\left\{-\frac{(h-x-st)^2}{4at}\right\} - \exp\left\{-\frac{(h-x)^2}{4at}\right\} \right) dh \\ &\quad + \lim_{t \rightarrow 0} \int_0^{+\infty} (4\pi at)^{-1/2} (-\Theta_0(h) + \theta_-) \left( \exp\left\{-\frac{(h+x-st)^2}{4at}\right\} - \exp\left\{-\frac{(h+x)^2}{4at}\right\} \right) dh. \end{aligned}$$

Use Lebesgue control theorem we know

$$\begin{aligned} &\lim_{t \rightarrow 0} \int_0^{+\infty} (4\pi at)^{-1/2} |\Theta_0(h) - \theta_-| \left| \exp\left\{-\frac{(h-x-st)^2}{4at}\right\} - \exp\left\{-\frac{(h-x)^2}{4at}\right\} \right| dh \\ &+ \lim_{t \rightarrow 0} \int_0^{+\infty} (4\pi at)^{-1/2} |-\Theta_0(h) + \theta_-| \left| \exp\left\{-\frac{(h+x-st)^2}{4at}\right\} - \exp\left\{-\frac{(h+x)^2}{4at}\right\} \right| dh \\ &\leq \lim_{t \rightarrow 0} C \int_0^{+\infty} \left| \exp\left\{-\frac{(h-x-st)^2}{4at}\right\} - \exp\left\{-\frac{(h-x)^2}{4at}\right\} \right| d(4at)^{-1/2} h \end{aligned}$$



$$\begin{aligned}
& + \lim_{t \rightarrow 0} C \int_0^{+\infty} \left| \exp\left\{-\frac{(h+x-st)^2}{4at}\right\} - \exp\left\{-\frac{(h+x)^2}{4at}\right\} \right| d(4at)^{-1/2} h \\
& \leq C \int_{-\infty}^{+\infty} e^{-\xi^2} \lim_{t \rightarrow 0} \left( \exp\left\{-\left(\xi - \frac{st}{\sqrt{4at}}\right)^2 + \xi^2\right\} - 1 \right) d\xi = 0,
\end{aligned}$$

which means

$$\lim_{t \rightarrow 0} (\theta_2(x, t) - \theta_{20}(x)) = 0.$$

So (2.2)<sub>3</sub> is established.

Now we consider the estimate about  $K$ . In fact from (2.6) we know

$$K = -\hat{I}_2 = \int_0^{+\infty} (4\pi at)^{-1/2} \Theta_{0z}(z) \exp\left\{-\frac{(z-st+y)^2}{4at}\right\} dz, \quad (2.7)$$

so we can get

$$\begin{aligned}
\int_0^t \int_0^{+\infty} |K| dx dt &= \int_0^t \int_0^{+\infty} |\hat{I}_2| dx dt \\
&\leq C \int_0^t (4\pi at)^{-1/2} \|\Theta_{0z}\| \left( \int_0^{+\infty} \exp\left\{-\frac{(z+x)^2}{4at}\right\} \exp\left\{-\frac{s^2 t}{4a}\right\} dz \right)^{1/2} dt \\
&\quad \times \int_0^{+\infty} (4\pi at)^{-1/2} \exp\left\{-\frac{(z+x)^2}{4at}\right\} dx \\
&\leq C \|\Theta_{0z}\| (1+t)^{1/2} \leq C \alpha^{1/2} \delta_0^{1/2} (1+t)^{1/2}.
\end{aligned}$$

And use Hölder inequality and Fubini Theorem and  $1 < |\theta_+ - \theta_-| \leq \|\Theta_{0z}\|_{L^1(\mathbb{R}_+)} < C$  (see Lemma 2.1), we can get

$$\begin{aligned}
\int_0^t \int_0^{+\infty} |\hat{I}_1|^2 dx dt &\leq C \int_0^t \int_0^{+\infty} (4\pi at)^{-1} \left( \int_0^{+\infty} \Theta_{0z} \exp\left\{-\frac{(z-st-x)^2}{4at}\right\} dz \right)^2 dx dt \\
&\leq C \int_0^t \int_0^{+\infty} (4\pi at)^{-1} \int_0^{+\infty} |\Theta_{0z}| \exp\left\{-\frac{(z-st-x)^2}{4at}\right\} dz dx \int_0^{+\infty} |\Theta_{0z}| dz dt \\
&\leq C \sqrt{1+t}.
\end{aligned}$$

In all when we combine with the estimates about  $\hat{I}_1$  and  $\hat{I}_2$  of (2.6) we can get

$$\int_0^t \int_0^{+\infty} \theta_{2x}^2 dx dt \leq C \sqrt{1+t}.$$

Now both side of (2.2)<sub>1</sub> multiply by  $\theta_{2xx}$ , integrate in  $\mathbb{R}_+ \times (0, t)$  and combine with Cauchy-Schwarz inequality we can get

$$\|\theta_{2x}\|^2 + \int_0^t \theta_{2x}^2(0, t) dt + \int_0^t \int_0^{+\infty} \theta_{2xx}^2 dx dt \leq C \|\Theta_{0x}\|^2 \leq C \alpha \delta_0. \quad (2.8)$$

So we finish this lemma.  $\square$

Now let's consider the time estimates about  $\partial_x^i \theta$  ( $i = 1, 2, 3$ ) of (1.8), we have the following results. We list the proof steps of each formula inside this lemma for reading convenient.

**Lemma 2.3** *If  $\Theta_{0x}$  satisfying the condition of Theorem 1.1, we can get*

$$\|(\ln \Theta)_x\|^2 + \int_0^t (\ln \Theta)_x^2(0, t) dx + a \int_0^t \|(\ln \Theta)_{xx}\|^2 dt \leq C\alpha\delta_0. \quad (2.9)$$

(see(2.19)–(2.21))

$$\|\Theta - \theta_2\|^2 + \int_0^t \|(\ln \Theta)_x\|^2 dt \leq C(1+t)^{1/2}. \quad (2.10)$$

(see(2.17)–(2.22))

$$\|(\ln \Theta)_x\|^2 \leq C(1+t)^{-1/2}. \quad (2.11)$$

(see(2.23)–(2.26))

$$\|(\ln \Theta)_{xx}\|^2 \leq C(1+t)^{-3/2}. \quad (2.12)$$

(see(2.27)–(2.33))

$$\begin{aligned} & \|(\ln \Theta)_{xx}\|^2(1+t) + \int_0^t \|\partial_x^3 \ln \Theta\|^2(1+t) dt \\ & + \int_0^t (\partial_x^2 \ln \Theta)^2(0, t)(1+t) dt \leq C\delta_0. \end{aligned} \quad (2.13)$$

(see(2.34))

$$\|\partial_x^3 \ln \Theta\|^2 \leq C(1+t)^{-5/2}. \quad (2.14)$$

(see(2.35)–(2.37))

$$\int_0^t (\partial_x^3 \ln \Theta)^2(0, t) dt \leq C. \quad (2.15)$$

(see(2.38))

$$\int_{\mathbb{R}_+} \Theta_x^2 dx \leq C\delta_0. \quad (2.16)$$

(see(2.39)–(2.40))

*Proof.*

Both side of (1.8)<sub>1</sub>–(2.2)<sub>1</sub> multiply by  $\Theta - \theta_2$ , integrate in  $\mathbb{R}_+ \times (0, t)$  and combine with Cauchy-Schwarz inequality we can get

$$\begin{aligned} & \|\Theta - \theta_2\|^2 + \int_0^t \|(\ln \Theta)_x\|^2 dt \\ & \leq C \int_0^t (\|\theta_{2x}\|^2 + \|K\|_{L^1}) dt + C \int_0^t ((\ln \Theta)_x^2 + \theta_{2x}^2)(0, t) dt. \end{aligned} \quad (2.17)$$

On the other hand from (1.8)<sub>1</sub>

$$(\ln \Theta)_t - s(\ln \Theta)_x = a \frac{(\ln \Theta)_{xx}}{\Theta}, \quad (2.18)$$

both side of it multiply by  $(\ln \Theta)_{xx}$  and integrate in  $\mathbb{R}_+ \times (0, t)$  we can get

$$\begin{aligned} & \|(\ln \Theta)_x\|^2 + \int_0^t (\ln \Theta)_x^2(0, t) dx + a \int_0^t \|(\ln \Theta)_{xx}\|^2 dt \\ & \leq C\|(\ln \Theta_0)_x\|^2 + \int_0^t (\ln \Theta)_t (\ln \Theta)_x|_0^{+\infty} dt. \end{aligned} \quad (2.19)$$

According to Lemma 2.1 we know that  $\|(\ln \Theta)_x\|^2 \leq C\alpha\delta_0$ , when combine with (2.19) and

$$\Theta_t(+\infty, t) = 0, \quad \Theta_t(0, t) = 0 \quad (2.20)$$

we can get

$$\|(\ln \Theta)_x\|^2 + \int_0^t (\ln \Theta)_x^2(0, t) dx + a \int_0^t \|(\ln \Theta)_{xx}\|^2 dt \leq C\alpha\delta_0. \quad (2.21)$$

Use (2.8), (2.3), (2.4) and (2.21) to (2.17) we can get

$$\|\Theta - \theta_2\|^2 + \int_0^t \|(\ln \Theta)_x\|^2 dt \leq C(1+t)^{1/2}. \quad (2.22)$$

That is (2.10).

Next, both side of (1.8)<sub>1</sub> multiply  $\Theta^{-1}(\ln \Theta)_{xx}(1+t)$  and integrate in  $\mathbb{R}_+ \times (0, t)$ , we can get

$$\begin{aligned} & \int_0^t (1+t)(\ln \Theta)_t(\ln \Theta)_x(0, t) dt + s/2 \int_0^t (\ln \Theta)_x^2(0, t)(1+t) dt \\ &= a \int_0^t \int_0^{+\infty} \frac{(\ln \Theta)_{xx}^2}{\Theta}(1+t) dx dt + \int_0^t \int_0^{+\infty} ((\ln \Theta)_x)_t (1+t) dx dt. \end{aligned} \quad (2.23)$$

Because

$$\int_0^t (1+t)(\ln \Theta)_t(\ln \Theta)_x(0, t) dt = 0, \quad (2.24)$$

we can get

$$\begin{aligned} & (1+t)\|(\ln \Theta)_x\|^2 + \int_0^t \int_0^{+\infty} (1+t)(\ln \Theta)_{xx}^2 dx dt + \int_0^t (\ln \Theta)_x^2(0, t)(1+t) dt \\ & \leq C\|\Theta_{0x}\|^2 + \int_0^t \int_0^{+\infty} (\ln \Theta)_x^2 dx dt. \end{aligned} \quad (2.25)$$

Combine with (2.22) we can get

$$\begin{aligned} & (1+t)\|(\ln \Theta)_x\|^2 + \int_0^t \int_0^{+\infty} (1+t)(\ln \Theta)_{xx}^2 dx dt + \int_0^t (\ln \Theta)_x^2(0, t)(1+t) dt \\ & \leq C(1+t)^{1/2}. \end{aligned} \quad (2.26)$$

That means  $\|(\ln \Theta)_x\|^2 \leq C(1+t)^{-1/2}$ , which is (2.11).

Again from (1.8)<sub>1</sub> we can get

$$(\ln \Theta)_{xt} - s(\ln \Theta)_{xx} = a \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right)_x. \quad (2.27)$$

Both side of (2.27) multiply  $\partial_x^3 \ln \Theta$  and get

$$((\ln \Theta)_{xt}(\ln \Theta)_{xx})_x - 1/2(\partial_x^2 \ln \Theta)_t - s/2(\partial_x^2 \ln \Theta)_x = a \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right)_x \partial_x^3 (\ln \Theta). \quad (2.28)$$

Take  $\Theta_t - s\Theta_x = (a \ln \Theta)_{xx}$  into  $((\ln \Theta)_{xt}(\ln \Theta)_{xx})_x$  we can get

$$((\ln \Theta)_{xt}(\ln \Theta)_{xx})_x (1+t)^2$$

$$\begin{aligned}
&= a^{-1} ((\ln \Theta)_{xt}(\Theta_t - s\Theta_x))_x (1+t)^2 \\
&= a^{-1} ((\ln \Theta)_{xt}(\Theta_t))_x (1+t)^2 - sa^{-1} ((\ln \Theta)_{xt}\Theta_x)_x (1+t)^2 \\
&= a^{-1} ((\ln \Theta)_{xt}(\Theta_t))_x (1+t)^2 - sa^{-1} \left( \left( \frac{\Theta_{xt}}{\Theta} - \frac{\Theta_t \Theta_x}{\Theta^2} \right) \Theta_x \right)_x (1+t)^2 \\
&= a^{-1} ((\ln \Theta)_{xt}(\Theta_t))_x (1+t)^2 - sa^{-1} \left( \frac{1}{2} \left( \frac{\Theta_x^2}{\Theta} \right)_t + \frac{1}{2} \frac{\Theta_x^2 \Theta_t}{\Theta^2} - \frac{\Theta_t \Theta_x^2}{\Theta^2} \right)_x (1+t)^2 \\
&= a^{-1} ((\ln \Theta)_{xt}(\Theta_t))_x (1+t)^2 - sa^{-1} / 2 \left( \left( \frac{\Theta_x^2}{\Theta} (1+t)^2 \right)_{tx} - 2 \left( \frac{\Theta_x^2}{\Theta} \right)_x (1+t) - \left( \frac{\Theta_x^2 \Theta_t}{\Theta^2} \right)_x (1+t)^2 \right).
\end{aligned}$$

Now both side of (2.28) multiply  $(1+t)^2$  then integrate in  $\mathbb{R}_+ \times (0, t)$  and combine with (2.20),  $\Theta_x(\infty, t) = 0$  and Cauchy-Schwarz inequality we get for a small  $\epsilon > 0$ ,

$$\begin{aligned}
&-s/(2a) \int_0^{+\infty} \left( \frac{\Theta_x^2}{\Theta} (1+t)^2 \right)_x dx + s/(2a) \int_0^{+\infty} \left( \frac{\Theta_{0x}^2}{\Theta_0} \right)_x dx - s/a \int_0^t \frac{\Theta_x^2}{\Theta}(0, t)(1+t) dt \\
&\geq -s/2 \int_0^t (\ln \Theta)_{xx}^2(0, t)(1+t)^2 dt + a \int_0^t \int_0^{+\infty} \frac{(\ln \Theta)_{xxx}^2}{\Theta} (1+t)^2 dx dt \\
&\quad - \epsilon \int_0^t \int_0^{+\infty} (1+t)^2 (\ln \Theta)_{xxx}^2 dx dt - Ca\epsilon^{-1} \int_0^t \int_0^{+\infty} (1+t)^2 (\ln \Theta)_{xx}^2 (\ln \Theta)_x^2 dx dt \\
&\quad + 1/2 \|(\ln \Theta)_{xx}\|^2 (1+t)^2 - 1/2 \|(\ln \Theta_0)_{xx}\|^2 - \int_0^t \|(\ln \Theta)_{xx}\|^2 (1+t) dx \\
&\geq -s/2 \int_0^t (\ln \Theta)_{xx}^2(0, t)(1+t)^2 dt + Ca \int_0^t \int_0^{+\infty} \frac{(\ln \Theta)_{xxx}^2}{\Theta} (1+t)^2 dx dt \\
&\quad - C\epsilon^{-1} a \int_0^t \int_0^{+\infty} (1+t)^2 \|(\ln \Theta)_{xx}\| \|(\ln \Theta)_{xxx}\| (\ln \Theta)_x^2 dx dt \\
&\quad + 1/2 \|(\ln \Theta)_{xx}\|^2 (1+t)^2 - 1/2 \|(\ln \Theta_0)_{xx}\|^2 - \int_0^t \|(\ln \Theta)_{xx}\|^2 (1+t) dx. \tag{2.29}
\end{aligned}$$

From Lemma 2.1 we know that

$$\frac{\Theta_{0x}^2}{\Theta_0}(0) \leq C\alpha^2 \delta_0^2, \tag{2.30}$$

$$\|(\ln \Theta_0)_{xx}\|^2 \leq C\alpha^3 \delta_0^2.$$

Combine with (2.26) we can get

$$\left| s/(2a) \int_0^{+\infty} \left( \frac{\Theta_{0x}^2}{\Theta_0} \right)_x dx - s/a \int_0^t \frac{\Theta_x^2}{\Theta}(0, t)(1+t) dt \right| \leq C(1+t)^{1/2}. \tag{2.31}$$

Take (2.26) (2.30) and (2.31) into (2.29) we can get

$$\begin{aligned}
&\|(\ln \Theta)_{xx}\|^2 (1+t)^2 + \int_0^t (1+t)^2 (\ln \Theta)_{xx}^2(0, t) dt + \int_0^t \int_0^{+\infty} (1+t)^2 (\ln \Theta)_{xxx}^2 dx dt \\
&\leq C(1+t)^{1/2}, \tag{2.32}
\end{aligned}$$

which also means

$$\|(\ln \Theta)_{xx}\|^2 \leq C(1+t)^{-3/2}, \tag{2.33}$$

and finish (2.12).

If both side of (2.28) multiply by  $(1+t)$ , similar as the proof of (2.32), when combine with (2.21) we can get

$$\|(\ln \Theta)_{xx}\|^2(1+t) + \int_0^t \int_0^{+\infty} (1+t)(\partial_x^3 \ln \Theta)^2 dx dt + \int_0^t (\partial_x^2 \ln \Theta)^2(0,t)(1+t) dt \leq C\delta_0, \quad (2.34)$$

which means (2.13).

From (2.27) we can get

$$\partial_t(\ln \Theta)_{xx} - s\partial_x(\ln \Theta)_{xx} = a\partial_x^2 \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right). \quad (2.35)$$

Similar as (2.28) we need to deal with the boundary term about  $((\ln \Theta)_{xxt}(\ln \Theta)_{xxx})_x$ .

Because combine with (2.18) and (2.27) we can get

$$\begin{aligned} & ((\ln \Theta)_{xxt}(\ln \Theta)_{xxx})_x \\ &= 1/s ((\ln \Theta)_{xxt}(\ln \Theta)_{xx}(\ln \Theta)_t)_x - a/s \left( (\ln \Theta)_{xxt} \frac{(\ln \Theta)_{xx}^2}{\Theta} \right)_x \\ & \quad + 1/a ((\ln \Theta)_{xxt}\Theta(\ln \Theta)_{xt})_x - s/a ((\ln \Theta)_{xxt}\Theta(\ln \Theta)_{xx})_x \\ &:= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

when both side of (2.35) multiply  $\partial_x^4 \ln \Theta(1+t)^3$  then integrate in  $\mathbb{R}_+ \times (0, t)$  we can get

$$\begin{aligned} & \int_0^t \int_0^{+\infty} (((\ln \Theta)_{xxt}(\ln \Theta)_{xxx})_x - s/2\partial_x(\ln \Theta)_{xx}^2)(1+t)^3 dx dt \\ &= \int_0^t \int_0^{+\infty} (I_1 + I_2 + I_3 + I_4 - s/2\partial_x(\ln \Theta)_{xxx}^2)(1+t)^3 dx dt \\ &= \int_0^t \int_0^{+\infty} a\partial_x^2 \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right) \partial_x^4 \ln \Theta(1+t)^3 dx dt \\ & \quad + \int_0^t \int_0^{+\infty} \frac{1}{2} ((\partial_x^3 \ln \Theta)^2)_t (1+t)^3 dx dt. \end{aligned} \quad (2.36)$$

To finish (2.36), similar as (2.32) using (2.26) and (2.32) we can get

$$\|\partial_x^3 \ln \Theta\|^2(1+t)^3 + \int_0^t (1+t)^3 \|\partial_x^4 \ln \Theta\|^2 dt + \int_0^t (\partial_x^3 \ln \Theta)^2(1+t)^3(0,t) dt \leq C(1+t)^{1/2}. \quad (2.37)$$

This means (2.14) finished.

When we change  $(1+t)^3$  to  $(1+t)^2$  and combine with (2.13), we can get

$$\|\partial_x^3 \ln \Theta\|^2(1+t)^2 + \int_0^t (1+t)^2 \|\partial_x^4 \ln \Theta\|^2 dt + \int_0^t (\partial_x^3 \ln \Theta)^2(1+t)^2(0,t) dt \leq C, \quad (2.38)$$

which finish (2.15).

Now both side of (2.18) multiply by  $(\ln \Theta)_{xx}(2s(\tau-t)+x)$  and integrate in  $[-2s(\tau-t), \infty) \times (0, t)$ , we can get

$$\sum_{i=1}^6 K_i = \int_0^t \int_{-2s(\tau-t)}^{+\infty} ((\ln \Theta)_\tau(\ln \Theta)_x(2s(\tau-t)+x))_x dx d\tau - 1/2 \int_0^t \int_{-2s(\tau-t)}^{+\infty} ((\ln \Theta)_x^2(2s(\tau-t)+x))_\tau dx d\tau$$

$$\begin{aligned}
& -a \int_0^t \int_{-2s(\tau-t)}^{+\infty} (\ln \Theta)_{xx} (\ln \Theta)_x \Theta^{-1} dx d\tau - s/2 \int_0^t \int_{-2s(\tau-t)}^{+\infty} ((\ln \Theta)_x^2(x + 2s(\tau - t)))_x dx d\tau \\
& + \frac{s}{2} \int_0^t \int_{-2s(\tau-t)}^{+\infty} (\ln \Theta)_x^2 dx d\tau - \int_0^t \int_{-2s(\tau-t)}^{+\infty} a (\ln \Theta)_{xx}^2(x + 2s(\tau - t)) \Theta^{-1} dx d\tau = 0.
\end{aligned} \tag{2.39}$$

Use Cauchy-Schwarz inequality

$$|K_3| \leq \frac{-s}{4} \int_0^t \int_{-2s(\tau-t)}^{+\infty} (\ln \Theta)_x^2 dx d\tau + C \int_0^t \int_{-2s(\tau-t)}^{+\infty} (\ln \Theta)_{xx}^2 dx d\tau,$$

then combine with (2.9) and  $s < 0$  we can get

$$|K_3| + K_5 \leq C \int_0^t \int_{-2s(\tau-t)}^{+\infty} (\ln \Theta)_{xx}^2 dx d\tau + \frac{s}{4} \int_0^t \int_{-2s(\tau-t)}^{+\infty} (\ln \Theta)_x^2 dx d\tau \leq C\delta_0 + \frac{s}{4} \int_0^t \int_{-2s(\tau-t)}^{+\infty} (\ln \Theta)_x^2 dx d\tau.$$

Use parabolic extreme value theory we know  $|(\ln \Theta)_{xx}| \leq C$ , so  $\lim_{x \rightarrow +\infty} \Theta_x^2(x + 2s(\tau - t)) = 0$ . When combine with the estimates from  $K_1$  to  $K_5$ , Lemma 2.1 and  $\lim_{x \rightarrow +\infty} \Theta_x^2(x + 2s(\tau - t)) = 0$ , (2.39) can be change to

$$\begin{aligned}
& \int_0^{+\infty} (\ln \Theta)_x^2 dx + \int_0^t \int_{-2s(\tau-t)}^{+\infty} (\ln \Theta)_{xx}^2(x + 2s(\tau - t)) dx d\tau \\
& - \frac{s}{4} \int_0^t \int_{-2s(\tau-t)}^{+\infty} (\ln \Theta)_x^2 dx d\tau \leq C\delta_0.
\end{aligned} \tag{2.40}$$

So we finish this lemma.  $\square$

The next lemma is concerned with the relations between the viscous continuity and the contact discontinuity. We shall show that as the heat conductivity  $k$  goes to zero,  $(V, U, \Theta)$  will approximate  $(\bar{V}, \bar{U}, \bar{\Theta})$  in  $L^p(\mathbb{R}_+)$  ( $p \geq 1$ ) norm on any finite time interval.

**Lemma 2.4** *For any given  $T \in (0, +\infty)$  independent of  $\kappa$  such that for any  $p \geq 1$  and  $t \in [0, T]$ ,*

$$\|(V - \bar{V}, U - \bar{U}, \Theta - \bar{\Theta})\|_{L^p(\mathbb{R}_+)} \rightarrow 0, \quad \text{as } \kappa \rightarrow 0.$$

*Proof.*

Letting

$$\Omega_1 = (0, -st) \quad \text{and} \quad \Omega_2 = (-st, +\infty).$$

By the definition of  $\bar{\Theta}$  in (1.5), to estimate  $\|\Theta - \bar{\Theta}\|_{L^p(\mathbb{R}_+)}$ , it suffices to prove

$$\|\Theta - \theta_-\|_{L^p(\Omega_1)}, \quad \|\Theta - \theta_+\|_{L^p(\Omega_2)} \rightarrow 0, \quad \text{as } \kappa \rightarrow 0, \quad p \geq 1.$$

Because

$$\|\Theta - \theta_-\|_{L^p(\Omega_1)}^p \leq C \|\Theta - \theta_-\|_{L^1(\Omega_1)}, \quad \|\Theta - \theta_+\|_{L^p(\Omega_2)}^p \leq C \|\Theta - \theta_+\|_{L^1(\Omega_2)},$$

the only thing we need to proof is

$$\lim_{\kappa \rightarrow 0} \|\Theta - \theta_-\|_{L^1(\Omega_1)} + \|\Theta - \theta_+\|_{L^1(\Omega_2)} = 0.$$

In fact we set  $sgn_\eta(l) = \begin{cases} 1, & l > \eta; \\ l/\eta, & -\eta \leq l \leq \eta; \\ -1, & l < -\eta. \end{cases}$ ,  $I_\eta(l) = \int_0^l sgn_\eta(l)dl$  and  $\eta > 0$ . Both side of (1.8)<sub>1</sub> multiply by  $sgn_\eta(\Theta - \theta_-)$  and integrate in  $(0, -s\tau) \times (0, t)$  we can get

$$\begin{aligned} \int_0^t \left( \int_0^{-s\tau} I_\eta(\Theta - \theta_-) dx \right)_\tau d\tau &= a \int_0^t (\ln \Theta)_x(-st, t) sgn_\eta(\Theta - \theta_-)(-st, t) d\tau \\ &\quad - a \int_0^t \int_0^{-s\tau} (\ln \Theta)_x^2 sgn'_\eta(\Theta - \theta_-) dx d\tau. \end{aligned}$$

So when  $\eta \rightarrow 0$  and use (2.9) and (2.13) we can get

$$\begin{aligned} \|\Theta - \theta_-\|_{L^1(\Omega_1)} + a \int_0^t \int_0^{-s\tau} (\ln \Theta)_x^2 sgn'_\eta(\Theta - \theta_-) dx d\tau \\ = a \int_0^t (\ln \Theta)_x(-st, t) sgn_\eta(\Theta - \theta_-)(-st, t) d\tau. \end{aligned} \quad (2.41)$$

Again, both side of (1.8)<sub>1</sub> multiply by  $sgn_\eta(\Theta - \theta_+)$  and integrate in  $(-s\tau, +\infty) \times (0, t)$  we can get

$$\begin{aligned} \int_0^t \left( \int_{-s\tau}^{+\infty} I_\eta(\Theta - \theta_+) dx \right)_\tau d\tau &= -a \int_0^t (\ln \Theta)_x(-st, t) sgn_\eta(\Theta - \theta_+)(-st, t) d\tau \\ &\quad - a \int_0^t \int_{-s\tau}^{+\infty} (\ln \Theta)_x^2 sgn'_\eta(\Theta - \theta_+) dx d\tau. \end{aligned}$$

When  $\eta \rightarrow 0$  and use (2.9), (2.13) and Lemma 2.1 we can get

$$\begin{aligned} \|\Theta - \theta_+\|_{L^1(\Omega_2)} + a \int_0^t \int_{-s\tau}^{+\infty} (\ln \Theta)_x^2 sgn'_\eta(\Theta - \theta_+) dx d\tau \\ = -a \int_0^t (\ln \Theta)_x(-st, t) sgn_\eta(\Theta - \theta_-)(-st, t) d\tau + \|\Theta_0 - \theta_+\|_{L^1(\Omega_2)}. \end{aligned} \quad (2.42)$$

Similar as (2.32), when we integrate (2.28) in  $\mathbb{R}_+ \times (0, t)$  and combine with (2.9) we can get there exist constant  $C > 0$  independent of  $\alpha$  such that

$$\|(\ln \Theta)_{xx}\|^2 + a \int_0^t \|\partial_x^3(\ln \Theta)\|^2 d\tau + \int_0^t (\ln \Theta)_{xx}^2(0, \tau) d\tau \leq Ca^{-1}\alpha^2 + C\alpha^3 + Ca\alpha^3 + Ca^{-1}\alpha. \quad (2.43)$$

Since  $a = \kappa p_+(\gamma - 1)/(\gamma R^2)$  and  $\kappa \rightarrow 0$ , we can choose  $\kappa = \alpha^{-1/2} < 1$ , use (2.9) and (2.43) such that (2.41) and (2.42) are meant

$$\|\Theta - \theta_-\|_{L^1(\Omega_1)} + \|\Theta - \theta_+\|_{L^1(\Omega_2)} \leq Ct(a\alpha + (a\alpha)^{3/4} + a^{5/4}\alpha) + C\alpha^{-1} \leq C\kappa^{3/8}(t + 1),$$

so we get  $\|(V - \bar{V}, \Theta - \bar{\Theta})\|_{L^p} \rightarrow 0$  as  $\kappa \rightarrow 0$  with any  $t \in [0, T]$ .

It remains to estimate  $\|U - \bar{U}\|_{L^p}$ . To do so, both side of (2.27) multiply by  $sgn_\eta((\ln \Theta)_x)$  then integrate in  $\mathbb{R}_+ \times (0, t)$  we can get

$$\int_0^t \left( \int_{\mathbb{R}_+} I_\eta((\ln \Theta)_x) dx \right)_\tau d\tau + s \int_0^t I_\eta((\ln \Theta)_x)(0, \tau) d\tau + a \int_0^t \int_{\mathbb{R}_+} \frac{(\ln \Theta)_{xx}^2}{\Theta} sgn'_\eta((\ln \Theta)_x) dx d\tau$$

$$= -a \int_0^t \theta_-^{-1} (\ln \Theta)_{xx}(0, \tau) \operatorname{sgn}_\eta((\ln \Theta)_x)(0, \tau) d\tau.$$

Again let  $\eta \rightarrow 0$ ,  $\kappa = \alpha^{-1/2} < 1$  we can get from (2.9), (2.43) and Lemma 2.1 that there exist constant  $C > 0$  independent of  $\alpha$  such that

$$\begin{aligned} & \int_{\mathbb{R}_+} |(\ln \Theta)_x| dx + a \int_0^t \int_{\mathbb{R}_+} \frac{(\ln \Theta)_{xx}^2}{\Theta} \operatorname{sgn}'_\eta((\ln \Theta)_x) dx d\tau \\ & \leq C t^{1/2} \left( \int_0^t (\ln \Theta)_{xx}^2(0, \tau) d\tau \right)^{1/2} + C t^{1/2} \left( \int_0^t (\ln \Theta)_x^2(0, \tau) d\tau \right)^{1/2} + \|(\ln \Theta)_x\|_{L^1} \\ & \leq C t^{1/2} (a^{-1/2} \alpha + \alpha^{3/2} + a^{1/2} \alpha^{3/2} + \alpha^{1/2}) + C \leq C(1+t)^{1/2} \alpha^2. \end{aligned}$$

Use the definition of  $U$  in (1.7) and combine with (2.9), (2.43) and  $\kappa = \alpha^{-1/2} < 1$  we know that

$$\begin{aligned} \|U - \bar{U}\|_{L^p}^p & \leq C \kappa^p \|(\ln \Theta)_x\|_{L^1} \|(\ln \Theta)_x\|^{(p-1)/2} \|(\ln \Theta)_{xx}\|^{(p-1)/2} \\ & \leq C (\alpha^{5(p-1)/4} + \alpha^{p-1} + \alpha^{(3-p)/2}) \alpha^{-2p} \alpha^2 (1+t)^{1/2} \\ & \leq C \left( \alpha^{-3(p-1)/4} + \alpha^{1-p} + \alpha^{-5(p-1)/2} \right) (1+t)^{1/2}. \end{aligned}$$

Remind that  $\alpha = \kappa^{-1/2}$ , so we can get

$$\lim_{\kappa \rightarrow 0} \|U - \bar{U}\|_{L^p} = 0.$$

The proof of Lemma 2.4 is therefore complete, which also means  $(V, U, \Theta)$  is viscous contact discontinuity.  $\square$

### 3 Proof of Theorem 1.1

Under the preparations in last section, the main task here is to finish (2.1). This part we also do some preparations. we must use the results

$$\begin{aligned} |V_x| & \leq C |\Theta_x|, \\ |\Theta_x|^2 & \leq C \|(\ln \Theta)_x\| \|(\ln \Theta)_{xx}\|, \\ |U_x| & \leq C |(\ln \Theta)_{xx}|, \\ |U_x|^2 & \leq C \|(\ln \Theta)_{xx}\| \|(\ln \Theta)_{xxx}\|, \end{aligned} \tag{3.1}$$

which come from (1.7)–(1.9). Also we set  $C(\delta_0)$  stands for small constants about  $\delta_0$ ,  $\|\varphi_0, \psi_0, \zeta_0\|$  is asked suitably small,  $C_v = \frac{R}{\gamma - 1}$  and

$$\epsilon_1 \ll \epsilon_3 \ll \epsilon_2.$$

Now, let's finish (2.1) which is very important for our proof of Theorem 1.1.

**Lemma 3.1** *If  $\epsilon_1 > 0$  and  $C(\delta_0) > 0$  are small constant about  $\delta_0$ , we can get*

$$\begin{aligned} & \int_{\mathbb{R}_+} \left( R\theta\Phi\left(\frac{v}{V}\right) + \frac{1}{2}\psi^2 + C_v\theta\Phi\left(\frac{\theta}{\Theta}\right) \right) dx + \int_0^t \left\| \left( \frac{\psi_x}{\sqrt{v\theta}}, \frac{\zeta_x}{\theta\sqrt{v}} \right) \right\|^2 d\tau \\ & \leq C\epsilon_1^{-1} \int_0^t \int_0^{+\infty} \Theta_x^2 (\varphi^2 + \zeta^2) dx d\tau + C(\delta_0) + C \left\{ \epsilon_1 \int_0^t (\|\varphi_x\|^2 + \psi_x^2(0, \tau)) d\tau + \|(\varphi_0, \psi_0, \zeta_0)\|^2 \right\}. \end{aligned}$$



*Proof.* Set

$$\begin{aligned}\Phi(z) &= z - \ln z - 1, \\ \Psi(z) &= z^{-1} + \ln z - 1,\end{aligned}$$

where  $\Phi'(1) = \Phi(1) = 0$  is a strictly convex function around  $z = 1$ . Similar to the proof in [4], we deduce from (1.12) that

$$\begin{aligned}& \left( \frac{\psi^2}{2} + R\Theta\Phi\left(\frac{v}{V}\right) + C_v\Theta\Phi\left(\frac{\theta}{\Theta}\right) \right)_t - s\left( \frac{\psi^2}{2} + R\Theta\Phi\left(\frac{v}{V}\right) + C_v\Theta\Phi\left(\frac{\theta}{\Theta}\right) \right)_x \\ & + \mu \frac{\Theta\psi_x^2}{v\theta} + \kappa \frac{\Theta\zeta_x^2}{v\theta^2} + H_x + Q = \mu \left( \frac{\psi\psi_x}{v} \right)_x - F\psi - \frac{\zeta G}{\theta},\end{aligned}\tag{3.2}$$

where

$$H = R \frac{\zeta\psi}{v} - R \frac{\Theta\varphi\psi}{vV} + \mu \frac{U_x\varphi\psi}{vV} - \kappa \frac{\zeta\zeta_x}{v\theta} + \kappa \frac{\Theta_x\varphi\zeta}{v\theta V},$$

and

$$\begin{aligned}Q &= p_+\Phi\left(\frac{V}{v}\right)U_x + \frac{p_+}{\gamma-1}\Phi\left(\frac{\Theta}{\theta}\right)U_x - \frac{\zeta}{\theta}(p_+-p)U_x - \mu \frac{U_x\varphi\psi_x}{vV} \\ & - \kappa \frac{\Theta_x}{v\theta^2}\zeta\zeta_x - \kappa \frac{\Theta\Theta_x}{v\theta^2V}\varphi\zeta_x - 2\mu \frac{U_x}{v\theta}\zeta\psi_x + \kappa \frac{\Theta_x^2}{v\theta^2V}\varphi\zeta + \mu \frac{U_x^2}{v\theta V}\varphi\zeta \\ & =: \sum_{i=1}^9 Q_i.\end{aligned}$$

Note that  $p = R\theta/v$ ,  $p_+ = R\Theta/V$  and (1.7), use integrate by part and Cauchy-Schwarz inequality can get

$$\begin{aligned}Q_1 + Q_2 &= Ra \left( \Phi\left(\frac{V}{v}\right)(\ln \Theta)_x \right)_x + \frac{Ra}{\gamma-1} \left( \Phi\left(\frac{\Theta}{\theta}\right)(\ln \Theta)_x \right)_x \\ & - aR(\ln \Theta)_x \left( \frac{V\varphi_x\varphi - V_x\varphi^2}{Vv^2} \right) \\ & - a \frac{p_+}{\gamma-1}(\ln \Theta)_x \left( \frac{\Theta\zeta_x\zeta - \Theta_x\zeta^2}{\Theta\theta^2} \right) \\ & \geq \left( p_+\Phi\left(\frac{V}{v}\right)U + \frac{p_+}{\gamma-1}\Phi\left(\frac{\Theta}{\theta}\right)U \right)_x \\ & - \epsilon(\zeta_x^2 + \varphi_x^2) - C\epsilon^{-1}\Theta_x^2(\zeta^2 + \varphi^2).\end{aligned}\tag{3.3}$$

Similarly, using  $p - p_+ = \frac{R\zeta - p_+\varphi}{v}$ , we can get

$$Q_3 \geq \frac{R\zeta - p_+\varphi}{v} \left( \frac{\zeta}{\theta} U_x \right) \geq \left( \frac{R\zeta^2 U}{v\theta} - \frac{p_+\zeta\varphi U}{\theta v} \right)_x - \epsilon(\zeta_x^2 + \varphi_x^2) - C\epsilon^{-1}\Theta_x^2(\zeta^2 + \varphi^2).\tag{3.4}$$

And

$$\begin{aligned}(Q_4 + Q_7) + (Q_5 + Q_6 + Q_8) + Q_9 &\geq -C\epsilon^{-1}(\ln \Theta)_{xx}^2 - \epsilon\psi_x^2 \\ & - \epsilon\zeta_x^2 - C\epsilon^{-1}\Theta_x^2(\zeta^2 + \varphi^2)\end{aligned}$$

$$-C\epsilon^{-1}|(\ln \Theta)_{xx}|^2(\zeta^2 + \varphi^2). \quad (3.5)$$

At the end we use the definition of  $F$  and  $G$  in (1.10) then combine with the general inequality skills as above to get

$$\begin{aligned} -F\psi - G\frac{\zeta}{\theta} &= -\frac{\kappa a(\gamma-1) - \mu p_+ \gamma}{R\gamma} \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right)_x \psi \\ &\quad + \frac{\mu p_+}{R\Theta} \left( \frac{\kappa(\gamma-1)}{R\gamma} (\ln \Theta)_{xx} \right)^2 \frac{\zeta}{\theta} \\ &\leq -\frac{\kappa a(\gamma-1) - \mu p_+ \gamma}{R\gamma} \left( \frac{(\ln \Theta)_{xx}}{\Theta} \psi \right)_x + \frac{\kappa a(\gamma-1) - \mu p_+ \gamma}{R\gamma} \frac{(\ln \Theta)_{xx}}{\Theta} \psi_x \\ &\quad + \frac{\mu p_+}{R\Theta} \left( \frac{\kappa(\gamma-1)}{R\gamma} (\ln \Theta)_{xx} \right)^2 \frac{\zeta}{\theta} \\ &\leq -\frac{\kappa a(\gamma-1) - \mu p_+ \gamma}{R\gamma} \left( \frac{(\ln \Theta)_{xx}}{\Theta} \psi \right)_x + \epsilon \psi_x^2 + C\epsilon^{-1} (\ln \Theta)_{xx}^2. \end{aligned} \quad (3.6)$$

Integrating (3.3)–(3.6) in  $\mathbb{R} \times (0, t)$ , using (2.9), (2.13) and the boundary condition about  $(\varphi, \psi, \zeta)$  of (1.12) to estimate the terms  $\mu \left( \frac{\psi \psi_x}{v} \right)_x$ ,  $\left( \frac{(\ln \Theta)_{xx} \psi}{\Theta} \right)_x$  and  $H_x$ , in the end combine with Cauchy-Schwarz inequality we now that for a small  $\epsilon > 0$  which is about  $\delta_0$ ,  $C_v = \frac{R}{\gamma-1}$ , we have

$$\begin{aligned} &\int_{\mathbb{R}_+} \left( R\theta \Phi \left( \frac{v}{V} \right) + \frac{1}{2} \psi^2 + C_v \theta \Phi \left( \frac{\theta}{\Theta} \right) \right) dx + \int_0^t \left\| \left( \psi_x / (\sqrt{v}\theta), \zeta_x / (\theta \sqrt{v}) \right) \right\|^2 d\tau \\ &\leq C\epsilon^{-1} \left\{ \int_0^t \int_0^{+\infty} \Theta_x^2 (\varphi^2 + \zeta^2) dx d\tau + \|\Theta_{0x}\|^2 \right\} + C \left\{ \epsilon \int_0^t \|\varphi_x\|^2 d\tau + \|(\varphi_0, \psi_0, \zeta_0)\|^2 \right\} \\ &\quad + C \int_0^t \psi^2(0, \tau) d\tau + \epsilon \int_0^t \psi_x^2(0, \tau) d\tau + C \int_0^t (\ln \Theta)_{xx}^2(0, \tau) d\tau + C(\delta_0). \end{aligned} \quad (3.7)$$

Using the definition about  $\psi(0, t)$  in (1.12), then combine with (1.9)<sub>5</sub> and (2.9), (2.13) we can get

$$\int_0^t \psi^2(0, \tau) d\tau + \int_0^t (\ln \Theta)_{xx}^2(0, \tau) d\tau \leq C(\delta_0), \quad (3.8)$$

Insert (3.8) into (3.7) we finish this lemma.  $\square$

**Lemma 3.2** *If  $C(\delta_0) > 0$  is a small constant about  $\delta_0$*

$$\int_0^t \int_{\mathbb{R}_+} \Theta_x^2 (\varphi^2 + \zeta^2) dx d\tau \leq C(\delta_0) \int_0^t \|(\varphi_x, \zeta_x)\|^2 d\tau.$$

*Proof.* Because if  $x > 0$

$$\begin{aligned} \frac{\varphi^2}{x+1} &= \int_0^x \left( \frac{2\varphi\varphi_x}{x+1} - \frac{\varphi^2}{(x+1)^2} \right) dx \\ &= \int_0^x \left( \varphi_x^2 - \left( \varphi_x - \frac{\varphi}{x+1} \right)^2 \right) dx \leq \int_0^x \varphi_x^2 dx \leq \|\varphi_x\|^2. \end{aligned}$$

similar as above we can get

$$\frac{\zeta^2}{x+1} \leq \int_0^x \zeta_x^2 dx \leq \|\zeta_x\|^2.$$

As to

$$\begin{aligned} \int_0^t \int_0^{+\infty} \Theta_x^2(\varphi^2 + \zeta^2) dx d\tau &\leq \int_0^t \int_0^{+\infty} \Theta_x^2(x+1) \frac{(\varphi^2 + \zeta^2)}{1+x} dx d\tau \\ &\leq \int_0^t \left( \int_0^{+\infty} \Theta_x^2(1+x) dx \right) \|(\varphi_x, \zeta_x)\|^2 d\tau, \end{aligned}$$

use (2.9) and (2.16) we can get

$$\int_0^t \int_{\mathbb{R}_+} \Theta_x^2(\varphi^2 + \zeta^2) dx d\tau \leq C(\delta_0) \int_0^t \|(\varphi_x, \zeta_x)\|^2 d\tau,$$

and we finish this lemma.  $\square$

**Lemma 3.3** *If a constant  $\epsilon_2 > 0$ ,  $C(\delta_0) > 0$  is a small constant about  $\delta_0$ , we can get*

$$\begin{aligned} &\|(\varphi, \psi, \zeta)\|^2 + \|(\psi_x, \zeta_x)\|^2 + \int_0^t (\psi_x^2(0, \tau) + \zeta_x^2(0, \tau)) d\tau + \int_0^t \|(\psi_{xx}, \zeta_{xx})\|^2 d\tau \\ &\leq C (\|(\psi_{0x}, \zeta_{0x})\|^2 + \epsilon_2^{-1} \|(\varphi_0, \psi_0, \zeta_0)\|^2) + C \epsilon_2^{-1} \int_0^t \|\varphi_x\|^2 d\tau + C(\delta_0). \end{aligned}$$

.

*Proof.* First to get the estimate of  $\|\psi_x(t)\|$ , multiply both side of (1.12)<sub>2</sub> to  $\psi_{xx}$  to get

$$\begin{aligned} &\left( \frac{\psi_x^2}{2} \right)_t + s \left( \frac{\psi_x^2}{2} \right)_x + \mu \frac{\psi_{xx}^2}{v} = \mu \frac{\psi_x v_x}{v^2} \psi_{xx} + \mu \left( \frac{U_x \varphi}{vV} \right)_x \psi_{xx} \\ &- R \left( \frac{\Theta \varphi}{vV} \right)_x \psi_{xx} + R \left( \frac{\zeta}{v} \right)_x \psi_{xx} + F \psi_{xx} + (\psi_t \psi_x)_x := \sum_{i=1}^6 I_i. \end{aligned}$$

use last inequality integrate in  $\mathbb{R}_+ \times (0, t)$  ( $s = -u_b/v_- < 0$ ) to get

$$\begin{aligned} &\|\psi_x(t)\|^2 + \int_0^t \psi_x^2(0, \tau) d\tau + \int_0^t \|\psi_{xx}(\tau)\|^2 d\tau \\ &\leq C \|\psi_{0x}\|^2 + C \sum_{i=1}^6 \left| \int_0^t \int_0^\infty I_i dx d\tau \right|. \end{aligned} \tag{3.9}$$

Now deal with  $\iint |I_i| dx d\tau$  in the right side of (3.9). Using  $\epsilon$  small and  $v = \varphi + V$ ,  $R\Theta/V = p_+$  and (2.11), (2.12), (3.1) to get

$$\begin{aligned} \int_0^t \int_0^{+\infty} |I_1| dx d\tau &\leq C \int_0^t \int_0^{+\infty} |V_x| |\psi_x| |\psi_{xx}| dx d\tau + C \int_0^t \int_0^{+\infty} |\varphi_x| |\psi_x| |\psi_{xx}| dx d\tau \\ &\leq C \int_0^t \|V_x\| \|\psi_x\|_{L^\infty} \|\psi_{xx}\| d\tau + C \int_0^t \|\psi_x\|_{L^\infty} \|\varphi_x\| \|\psi_{xx}\| d\tau \end{aligned}$$

$$\begin{aligned}
&\leq \epsilon \int_0^t \|\psi_{xx}\|^2 d\tau + C\epsilon^{-1} \int_0^t \|\psi_x\|^2 \|V_x\|^4 d\tau + C \int_0^t \|\psi_x\|^{1/2} \|\varphi_x\| \|\psi_{xx}\|^{3/2} d\tau \\
&\leq C\epsilon \int_0^t \|\psi_{xx}\|^2 d\tau + C(\delta_0) \int_0^t \|\psi_x\|^2 d\tau \\
&\quad + C\epsilon^{-1} \sup_t \|\varphi_x\|^4 \int_0^t \|\psi_x\|^2 d\tau.
\end{aligned} \tag{3.10}$$

Next we use the definition of  $(V, U, \Theta)$  in (1.7), (1.9), Cauchy-Schwarz inequality and (2.13), (2.14), (3.1) to get

$$\begin{aligned}
&\int_0^t \int_0^\infty |I_2| dx d\tau \\
&\leq C \int_0^t \int_0^\infty (|U_{xx}||\varphi| + |U_x||\varphi_x| + |U_x||V_x||\varphi| + |U_x||\varphi||\varphi_x|) |\psi_{xx}| dx d\tau \\
&\leq \epsilon \int_0^t \|\psi_{xx}\|^2 d\tau + \frac{C}{\epsilon} \int_0^t \|\varphi\|_{L^\infty}^2 \|U_{xx}\|^2 d\tau + \frac{C}{\epsilon} \int_0^t \|U_x\|_{L^\infty}^2 \|\varphi_x\|^2 d\tau \\
&\quad + \frac{C}{\epsilon} \int_0^t \|\varphi\|_{L^\infty}^2 \|V_x\|^2 \|U_x\|_{L^\infty}^2 d\tau + \frac{C}{\epsilon} \int_0^t \|\varphi\|_{L^\infty}^2 \|U_x\|_{L^\infty}^2 \|\varphi_x\|^2 d\tau \\
&\leq \epsilon \int_0^t \|\psi_{xx}\|^2 d\tau + C(\delta_0) + C(\delta_0) \int_0^t \|\varphi_x\|^2 d\tau.
\end{aligned} \tag{3.11}$$

The same as (3.10) and (3.11), we use Lemma 3.2, the definition of  $F$  in (2.2) and (2.9), (2.11)–(2.13), (3.1) we can get the estimates about  $I_3$  to  $I_5$  as following.

$$\begin{aligned}
&\int_0^t \int_0^\infty (|I_3| + |I_4| + |I_5|) dx d\tau \\
&\leq C \int_0^t \int_0^\infty (|\Theta_x||\varphi| + |\Theta||\varphi_x| + |\Theta||V_x||\varphi| + |\Theta||\varphi||\varphi_x|) |\psi_{xx}| dx d\tau \\
&\quad + C \int_0^t \int_0^\infty (|\zeta_x| + |\zeta||V_x| + |\zeta||\varphi_x|) |\psi_{xx}| dx d\tau \\
&\quad + \epsilon \int_0^t \|\psi_{xx}\|^2 d\tau + \frac{C}{\epsilon} \int_0^t \|F\|^2 d\tau \\
&\leq \epsilon \int_0^t \|\psi_{xx}\|^2 d\tau + C\epsilon^{-1} \int_0^t \|\varphi_x\|^2 d\tau + \frac{C}{\epsilon} \int_0^t \|\varphi_x\|^2 d\tau + \frac{C}{\epsilon} \int_0^t \int_0^\infty V_x^2 \varphi^2 dx d\tau \\
&\quad + \epsilon \int_0^t \|\psi_{xx}\|^2 d\tau + \frac{C}{\epsilon} \int_0^t \int_0^\infty (\zeta_x^2 + V_x^2 \zeta^2) dx d\tau + \frac{C}{\epsilon} \sup_t \|(\varphi, \zeta)\| \|(\varphi_x, \zeta_x)\| \int_0^t \|\varphi_x\|^2 d\tau \\
&\quad + \epsilon \int_0^t \|\psi_{xx}\|^2 d\tau + C(\delta_0).
\end{aligned} \tag{3.12}$$

Because Lemma 3.1 and Lemma 3.2, we know  $\|(\varphi, \zeta)\|$  is suitably small when  $C(\delta_0)$  and  $\|(\varphi_0, \psi_0, \zeta_0)\|$  small. So there exist a small constant  $\delta$  about  $\|(\varphi_0, \psi_0, \zeta_0)\|$  and  $\delta_0$  such that

$$\frac{C}{\epsilon} \sup_t \|(\varphi, \zeta)\| \|(\varphi_x, \zeta_x)\| \int_0^t \|\varphi_x\|^2 d\tau \leq C\delta \int_0^t \|\varphi_x\|^2 d\tau + C\delta \int_0^t \psi_x^2(0, \tau) d\tau,$$

here  $\|\varphi_x\|^2 + \int_0^t \|\varphi_x\|^2 d\tau \leq C$  can be established in Lemma 3.4.

Therefore

$$\begin{aligned} & \int_0^t \int_0^\infty (|I_3| + |I_4| + |I_5|) dx d\tau \\ & \leq \epsilon \int_0^t \|\psi_{xx}\|^2 d\tau + \frac{C}{\epsilon} \int_0^t \int_0^\infty (\zeta_x^2 + \varphi_x^2) dx d\tau + C\delta \int_0^t \psi_x^2(0, \tau) d\tau + C(\delta_0). \end{aligned}$$

At last we integrate by part to the term about  $I_6$  to get

$$\begin{aligned} & \left| \int_0^t \int_0^\infty I_6 dx d\tau \right| = \left| \int_0^t (\psi_t \psi_x)(0, \tau) d\tau \right| \leq \frac{C}{C^{1/2}(\delta_0)} \int_0^t \psi_x^2(0, \tau) d\tau + C^{1/2}(\delta_0) \int_0^t \psi_\tau^2(0, \tau) d\tau \\ & \leq \frac{C}{\epsilon} C^{-1/2}(\delta_0) \int_0^t \|\psi_x\|^2 d\tau + \epsilon \int_0^t \|\psi_{xx}\|^2 d\tau + C^{1/2}(\delta_0) \int_0^t \psi_\tau^2(0, \tau) d\tau. \end{aligned} \quad (3.13)$$

Using the definition of  $U$  in (1.7),  $\psi = u - U$  and (2.27) to get

$$\begin{aligned} \psi_t(0, t) &= -\frac{k(\gamma-1)}{\gamma R} (\ln \Theta)_{xt}(0, t) \\ &= -s \frac{k(\gamma-1)}{\gamma R} (\ln \Theta)_{xx}(0, t) - a \frac{k(\gamma-1)}{\gamma R} \partial_x \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right) (0, t). \end{aligned} \quad (3.14)$$

Combine with (2.13) (2.15) and  $|\Theta_x(0, t)| \leq C$  we get

$$\int_0^t \|(\ln \Theta)_{xt}\|_{L^\infty}^2(0, \tau) d\tau \leq C. \quad (3.15)$$

So combine with (3.13) and (3.15) we get

$$\begin{aligned} & \left| \int_0^t \int_0^\infty I_6 dx d\tau \right| \\ & \leq \frac{C}{\epsilon} \int_0^t \|\psi_x\|^2 d\tau + \epsilon \int_0^t \|\psi_{xx}\|^2 d\tau + C(\delta_0). \end{aligned} \quad (3.16)$$

In all there exist a small  $\delta > 0$

$$\begin{aligned} & \int_0^t \int_0^{+\infty} \sum_{i=1}^6 |I_i| dx d\tau \\ & \leq C \int_0^t (\epsilon \|\psi_{xx}\|^2 + \delta \psi_x^2(0, \tau)) d\tau + CN^4(t) \epsilon^{-1} \int_0^t \|\psi_x\|^2 d\tau \\ & \quad + C\epsilon^{-1} \int_0^t \|(\varphi_x, \psi_x, \zeta_x)\|^2 d\tau + C(\delta_0). \end{aligned} \quad (3.17)$$

So (3.9) can be change to

$$\begin{aligned} & \|\psi_x(t)\|^2 + \int_0^t \psi_x^2(0, \tau) d\tau + \int_0^t \|\psi_{xx}(\tau)\|^2 d\tau \\ & \leq C(\delta_0) + C\epsilon^{-1} \int_0^t \|(\varphi_x, \psi_x, \zeta_x)\|^2 d\tau + CN^4(t) \epsilon^{-1} \int_0^t \|\psi_x\|^2 d\tau + C\|\psi_{0x}\|^2. \end{aligned} \quad (3.18)$$

The estimate about  $\|\zeta_x\|$  is similar to  $\|\psi_x\|$ , use (1.12)<sub>3</sub> multiply  $\zeta_{xx}$  then integrate in  $Q_t = \mathbb{R}_+ \times (0, t)$  to get

$$\begin{aligned}
& \|\zeta_x\|^2 + \int_0^t \zeta_x^2(0, \tau) d\tau + \int_0^t \|\zeta_{xx}\|^2 d\tau \\
& \leq C\|\zeta_{0x}\|^2 + C\epsilon^{-1} \int_0^t \int_0^\infty (\psi_x^2 + \zeta\psi_x^2 + \zeta^2 U_x^2 + U_x^2 \varphi^2) dx d\tau \\
& \quad + C \int_0^t \int_0^{+\infty} |\zeta_x|(|\varphi_x| + |V_x|)|\zeta_{xx}| dx d\tau + C\epsilon^{-1} \int_0^t \int_0^\infty \left| \left( \frac{\Theta_x \varphi}{vV} \right)_x \right|^2 dx d\tau \\
& \quad + C\epsilon^{-1} \int_0^t \int_0^{+\infty} (U_x^4 + \psi_x^4) dx d\tau + C\epsilon^{-1} \int_0^t \|G\|^2 d\tau \\
& =: C\|\zeta_{0x}\|^2 + \sum_{i=1}^5 J_i.
\end{aligned} \tag{3.19}$$

Use the same method as (3.10)–(3.13)

$$J_1 \leq C\epsilon^{-1} \int_0^t \|\psi_x\|^2 d\tau + C\epsilon^{-1} N^2(t) \int_0^t \|U_x\|^2 d\tau \leq C\epsilon^{-1} \int_0^t \|\psi_x\|^2 d\tau + C(\delta_0).$$

Again use the same method as (3.10)–(3.13)

$$\begin{aligned}
J_2 & \leq C \int_0^t \|\zeta_x\|_{L^\infty} \|\varphi_x\| \|\zeta_{xx}\| d\tau + C \int_0^t \|V_x\| \|\zeta_x\|_{L^\infty} \|\zeta_{xx}\| d\tau \\
& \leq C \int_0^t \|\zeta_x\|^{1/2} \|\zeta_{xx}\|^{3/2} \|\varphi_x\| d\tau + \epsilon \int_0^t \|\zeta_{xx}\|^2 d\tau + C(\delta_0) \int_0^t \|\zeta_x\|^2 d\tau \\
& \leq 2\epsilon \int_0^t \|\zeta_{xx}\|^2 d\tau + C(\delta_0) \int_0^t \|\zeta_x\|^2 d\tau + C\epsilon^{-1} \sup_t \|\varphi_x\|^4 \int_0^t \|\zeta_x\|^2 d\tau.
\end{aligned}$$

Because

$$\begin{aligned}
& \left| \left( \frac{\Theta_x \varphi}{vV} \right)_x \right|^2 \\
& = \left| \frac{\Theta_{xx} \varphi}{vV} + \frac{\Theta_x \varphi_x}{vV} + \frac{\Theta_x \varphi}{vV} \left( -\frac{V_x + \varphi_x}{v^2} - \frac{V_x}{V^2} \right) \right|^2 \\
& \leq C\Theta_{xx}^2 \varphi^2 + C\Theta_x^2 \varphi_x^2 + C\Theta_x^2 V_x^2 \varphi^2 + C\Theta_x^2 \varphi^2 \varphi_x^2,
\end{aligned}$$

combine with  $R\Theta/V = p_+$ , use the same method as (3.10)–(3.13) to get

$$\begin{aligned}
J_3 & \leq C\epsilon^{-1} \int_0^t \|\varphi\|_{L^\infty}^2 \|\Theta_{xx}\|^2 d\tau + C\epsilon^{-1} \int_0^t \|\Theta_x\|_{L^\infty}^2 \|\varphi_x\|^2 d\tau \\
& \quad + C\epsilon^{-1} \int_0^t \int_0^{+\infty} \Theta_x^2 V_x^2 \varphi^2 dx d\tau \\
& \leq C(\delta_0) \int_0^t \|\varphi_x\|^2 d\tau + C(\delta_0).
\end{aligned}$$

Use the definition  $U$  and similar as (3.10) (3.11) that we combine with Lemma 2.3 to get

$$J_4 \leq C(\delta_0) + C\epsilon^{-1} \int_0^t \|\psi_x\|_{L^\infty}^2 \|\psi_x\|^2 d\tau$$

$$\begin{aligned}
&\leq C(\delta_0) + C\epsilon^{-1} \int_0^t \|\psi_x\|^3 \|\psi_{xx}\| d\tau \\
&\leq C(\delta_0) + C \int_0^t (\epsilon^{-2} \|\psi_x\|^2 \|\psi_x\|^4 + \epsilon^2 \|\psi_{xx}\|^2) d\tau.
\end{aligned}$$

Use the definition  $G$  in (2.2) combine with Lemma 2.3

$$J_5 = C\epsilon^{-1} \int_0^t \|G\|^2 d\tau \leq C(\delta_0).$$

Use the results from  $J_1$  to  $J_5$ , the inequality (3.19) can be change to

$$\begin{aligned}
&\|\zeta_x\|^2 + \int_0^t \zeta_x^2(0, \tau) d\tau + \int_0^t \|\zeta_{xx}\|^2 d\tau \\
&\leq C\|\zeta_{0x}\|^2 + C(\epsilon^{-3} + N^4(t)) \int_0^t \|(\psi_x, \zeta_x)\|^2 d\tau + C(\delta_0) \int_0^t \|\varphi_x\|^2 d\tau + C(\delta_0) \\
&\quad + C\epsilon \int_0^t \|\psi_{xx}\|^2 d\tau.
\end{aligned} \tag{3.20}$$

In fact when combine with Lemma 3.1–3.2, (3.18) and (3.20), it is easy to get

$$\begin{aligned}
&\|(\varphi, \psi, \zeta)\|^2 + \|(\psi_x, \zeta_x)\|^2 + \int_0^t (\psi_x^2(0, \tau) + \zeta_x^2(0, \tau)) d\tau + \int_0^t \|(\psi_{xx}, \zeta_{xx})\|^2 d\tau \\
&\leq C(\|(\psi_{0x}, \zeta_{0x})\|^2 + \epsilon^{-3} \|(\varphi_0, \psi_0, \zeta_0)\|^2) + C\epsilon^{-1} \int_0^t \|\varphi_x\|^2 d\tau + C(\delta_0).
\end{aligned}$$

□

**Lemma 3.4** For a small  $\epsilon_3 > 0$  and  $C(\delta_0) > 0$  is a small constant about  $\delta_0$ , we can get

$$\begin{aligned}
\|\varphi_x\|^2 + \int_0^t \|\varphi_x\|^2 d\tau &\leq C\|\varphi_{0x}\|^2 + C\epsilon_3^{-1} \|(\varphi_0, \psi_0, \zeta_0)\|^2 + \int_0^t \epsilon_3 \|\psi_{xx}\|^2 d\tau \\
&\quad + \int_0^t C\epsilon_3^{-1} \|(\psi_x, \zeta_x)\|^2 d\tau + C(\delta_0).
\end{aligned} \tag{3.21}$$

*Proof.* Set  $\bar{v} = \frac{v}{V}$  take it into (1.12)<sub>1</sub>, (1.12)<sub>2</sub> ( $p = R\theta/v$ ) to get

$$\psi_t - s\psi_x + p_x = \mu \left( \frac{\bar{v}_x}{\bar{v}} \right)_t - s\mu \left( \frac{\bar{v}_x}{\bar{v}} \right)_x - F,$$

Both sides of last equation multiply  $\bar{v}_x/\bar{v}$  to get

$$\begin{aligned}
&\left( \frac{\mu}{2} \left( \frac{\bar{v}_x}{\bar{v}} \right)^2 - \psi \frac{\bar{v}_x}{\bar{v}} \right)_t + \frac{R\theta}{v} \left( \frac{\bar{v}_x}{\bar{v}} \right)^2 + \left( \psi \frac{\bar{v}_t}{\bar{v}} \right)_x - s\mu \left( \frac{\bar{v}_x}{\bar{v}} \right)_x \frac{\bar{v}_x}{\bar{v}} \\
&= \frac{\psi_x^2}{v} + U_x \left( \frac{1}{v} - \frac{1}{V} \right) \psi_x + \frac{R\zeta_x}{v} \frac{\bar{v}_x}{\bar{v}} - \frac{R\theta}{v} \left( \frac{1}{\Theta} - \frac{1}{\theta} \right) \Theta_x \frac{\bar{v}_x}{\bar{v}} + F \frac{\bar{v}_x}{\bar{v}}.
\end{aligned} \tag{3.22}$$

Because  $v|_{x=0} = V|_{x=0} = v_-$ , we can get

$$\left( \frac{\bar{v}_x}{\bar{v}} \right)^2(0, t) = \left( \frac{v_x}{v} - \frac{V_x}{V} \right)^2(0, t) = \frac{1}{s^2} \left( \frac{u_x}{v_-} - \frac{U_x}{v_-} \right)^2(0, t) = \frac{\psi_x^2(0, t)}{s^2 v_-^2}.$$

Use Cauchy-Schwarz inequality to get

$$\int_0^t \left( \frac{\bar{v}_x}{\bar{v}} \right)^2 (0, \tau) d\tau \leq C \int_0^t \psi_x^2(0, \tau) d\tau \leq C \int_0^t (\epsilon^{-1} \|\psi_x\|^2 + \epsilon \|\psi_{xx}\|^2) d\tau. \quad (3.23)$$

On the other hand if we integrate (3.22) in  $R_+ \times (0, t)$ , (3.22) is changed to

$$\begin{aligned} & \int_{\mathbb{R}_+} \left( \frac{\mu}{2} \left( \frac{\bar{v}_x}{\bar{v}} \right)^2 - \psi \frac{\bar{v}_x}{\bar{v}} \right) dx - \int_{\mathbb{R}_+} \left( \frac{\mu}{2} \left( \frac{\bar{v}_x(x, 0)}{\bar{v}(x, 0)} \right)^2 - \psi_0 \frac{\bar{v}_x(x, 0)}{\bar{v}(x, 0)} \right) dx \\ & + \int_0^t \int_{\mathbb{R}_+} \left( \frac{R\theta}{v} \left( \frac{\bar{v}_x}{\bar{v}} \right)^2 + \left( \psi \frac{\bar{v}_t}{\bar{v}} \right)_x - s\mu \left( \frac{\bar{v}_x}{\bar{v}} \right)_x \frac{\bar{v}_x}{\bar{v}} \right) dx d\tau \\ & \leq C\epsilon^{-1} \left( \int_0^t \|(\zeta_x, \psi_x)\|^2 d\tau + \int_0^t \int_0^{+\infty} \Theta_x^2(\varphi^2 + \zeta^2) dx d\tau \right) \\ & + C\epsilon^{-1} \int_0^t \int_0^{+\infty} U_x^2 \varphi^2 dx d\tau + C\epsilon^{-1} \int_0^t \int_0^{+\infty} |F|^2 dx d\tau + \epsilon \int_0^t (\|\frac{\bar{v}_x}{\bar{v}}\|^2 + \|\psi_{xx}\|^2) d\tau. \end{aligned}$$

Furthermore (3.22) can be change to the following inequality

$$\begin{aligned} & \int_0^t \|\frac{\bar{v}_x}{\bar{v}}\|^2 d\tau + \|\frac{\bar{v}_x}{\bar{v}}\|^2 - C\epsilon^{-1} \|\psi\|^2 - C\|\psi_0\|^2 - C \int_0^{+\infty} \frac{\bar{v}_x}{\bar{v}}(x, 0)^2 dx \\ & \leq C\epsilon^{-1} \left( \int_0^t \|(\zeta_x, \psi_x)\|^2 d\tau + \int_0^t \int_0^{+\infty} \Theta_x^2(\varphi^2 + \zeta^2) dx d\tau + C(\delta_0) \right) \\ & + \epsilon \int_0^t \left( \|\frac{\bar{v}_x}{\bar{v}}\|^2 + \|\psi_{xx}\|^2 \right) d\tau + C\|\varphi_{0x}\|^2. \end{aligned} \quad (3.24)$$

Because  $C_1(\varphi_x^2) - C_2V_x^2 \leq (\frac{\bar{v}_x}{\bar{v}})^2 \leq C_3\varphi_x^2 + C_4V_x^2$  ( $C_1, C_2, C_3, C_4$  stands for constants about  $v$ ), combine with Lemma 3.1–3.2 we can find a small  $\epsilon$  such that we change (3.24) to

$$\begin{aligned} \int_0^t \|\varphi_x\|^2 d\tau + \|\varphi_x\|^2 & \leq C\|\varphi_{0x}\|^2 + C\epsilon^{-1} \|(\varphi_0, \psi_0, \zeta_0)\|^2 + \int_0^t \epsilon \|\psi_{xx}\|^2 d\tau \\ & + \int_0^t C\epsilon^{-1} \|(\psi_x, \zeta_x)\|^2 d\tau + C(\delta_0). \end{aligned} \quad (3.25)$$

So we finish this lemma.  $\square$

From Lemma 3.1 to Lemma 3.4 we know when  $\delta_0$  and  $\|(\varphi_0, \psi_0, \zeta_0)\|$  suitably small there exist a suitably small positive constant  $\delta$  such that

$$\|(\varphi, \psi, \zeta)\|^2 + \int_0^t \|(\psi_x, \zeta_x)\|^2 d\tau \leq C\delta,$$

and

$$\|(\varphi_x, \psi_x, \zeta_x)\|^2 + \int_0^t \|(\psi_{xx}, \zeta_{xx})\|^2 d\tau \leq C.$$

Then we can get  $C_5 \leq |v| \leq C_6$  and  $C_7 \leq |\theta| \leq C_8$  when  $\delta$  small, here  $C_5, C_6, C_7$  and  $C_8$  are constants independent of  $v$  and  $\theta$ . When combine with Lemma 3.1–3.4 we can get (2.1) in Proposition 2.2.

To finish Theorem 1.1 now we will proof  $\sup_{x \in \mathbb{R}_+} |(\varphi, \psi, \zeta)| \rightarrow 0$ , as  $t \rightarrow \infty$ .



Because  $\int_0^{+\infty} \partial_x(1.12)_1 \times 2\varphi_x dx$  equals to

$$s\varphi_x^2(0, t) = 2 \int_0^\infty \varphi_x \psi_{xx} dx - \frac{d}{dt} \|\varphi_x\|^2, \quad (3.26)$$

use Cauchy-Schwarz inequality we get

$$2 \int_0^\infty \varphi_x \psi_{xx} dx \leq C (\|\varphi_x\|^2 + \|\psi_{xx}\|^2),$$

according to Lemma 3.3–3.4 and (3.23) to get

$$\int_0^\infty \varphi_x^2(0, t) dt \leq C\epsilon^{-1} (C(\delta_0) + \|(\varphi_0, \psi_0, \zeta_0)\|_1^2) + C\epsilon, \quad (3.27)$$

again using Lemma 3.3–3.4 and (3.26), then from (3.27) we get

$$\begin{aligned} & \int_0^\infty \left| \frac{d}{dt} \|\varphi_x(t)\|^2 \right| dt \\ & \leq C \int_0^\infty \varphi_x^2(0, t) dt + C \int_0^\infty (\|\varphi_x\|^2 + \|\psi_{xx}\|^2) dt \\ & \leq C\epsilon^{-1} (C(\delta_0) + \|(\varphi_0, \psi_0, \zeta_0)\|_1^2) + C\epsilon. \end{aligned} \quad (3.28)$$

Similar as above, from Lemma 3.1–3.4 and combine with Sobolev inequality we get

$$\int_0^\infty \left( \left| \frac{d}{dt} \|\psi_x(t)\|^2 \right| + \left| \frac{d}{dt} \|\zeta_x(t)\|^2 \right| \right) d\tau \leq C\epsilon_4^{-1} (C(\delta_0) + \|(\varphi_0, \psi_0, \zeta_0)\|_1^2) + C\epsilon. \quad (3.29)$$

It means

$$\|(\varphi, \psi, \zeta)(t)\|_{L^\infty}^2 \leq 2\|(\varphi, \psi, \zeta)(t)\| \|(\varphi_x, \psi_x, \zeta_x)(t)\| \rightarrow 0 \quad \text{when } t \rightarrow \infty.$$

So we finish Theorem 1.1.  $\square$

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